ISRAEL JOURNAL OF MATHEMATICS 131 (2002), 203–220

DECAY OF CORRELATIONS FOR PIECEWISE INVERTIBLE MAPS IN HIGHER DIMENSIONS

BY

JÉRÔME BUZZI

CMAT U.M.R. 7640 du C.N.R.S., Ecole polytechnique
91128 Palaiseau Cedex, France
e-mail: buzzi@math.polytechnique.fr

AND

VÉRONIQUE MAUME-DESCHAMPS

Laboratoire de topologie, U.M.R. 5584 du C.N.R.S., B.P. 47 870
21078 Dijon Cedex, France
e-mail: vmaume@u-bourgogne.fr

ABSTRACT

We study the mixing properties of equilibrium states μ of non-Markov piecewise invertible maps $T : X \to X$, especially in the multidimensional case. Assuming mainly Hölder continuity and that the topological pressure of the boundary is smaller than the total topological pressure, we establish exponential decay of correlations, i.e.,

$$\left| \int_X \varphi \cdot \psi \circ T^n \, d\mu - \int_X \varphi \, d\mu \cdot \int_X \psi \, d\mu \right| \leq C \cdot e^{-\alpha n}$$

for all Hölder functions $\varphi, \psi : X \to \mathbb{R}$, all $n \geq 0$ and some $C < \infty$, $\alpha > 0$. We also obtain a Central Limit Theorem. Weakening the smoothness assumption, we get subexponential rates of decay.

Received December 6, 1999 and in revised form April 2, 2001

0. Introduction

Our goal is to generalize ergodic and statistical properties of equilibrium states, which are well-known in dimension one [LY, HK, R, Kel, Sc, Go], to a natural *multidimensional* setting. Several questions in this regard have already been considered (existence and characterization of equilibrium states [Bu2], construction of conformal measures [BPS], absolutely continuous invariant measures [Bu0, Bu1, Bu3, Co, GB, Sau] or zeta functions [BuKe]), sometimes with surprising results [Bu4, T3].

In this paper we study the speed of mixing of equilibrium states and prove that it is exponential. This implies, e.g., the Central Limit Theorem.

We first consider Hölder continuous weights for simplicity and then move on to the more general case of summable moduli of continuity [Sc] (which was the setting for the previous works [Bu3, BPS]) relying on an abstract result proved in a companion paper [BuMa].

Our setting will be the following. $(X, \mathcal{Z}, T, g)$ will be a *weighted piecewise invertible map*, i.e.:

- $X = \overline{\bigcup_{Z\in\mathcal{Z}} Z}$ is a locally connected compact metric space.
- $\mathcal{Z}$ is a finite collection of pairwise disjoint, bounded and open subsets of X. Let $Y = \bigcup_{Z\in\mathcal{Z}} Z$.
- $T: Y \to X$ is a map such that each restriction $T|Z$, $Z \in \mathcal{Z}$, coincides with the restriction of a homeomorphism $T_Z: U \to V$ with U, V open sets such that $U \supset \bar{Z}$, $V \supset \overline{T(Z)}$.
- $g: X \to \mathbb{R}$.

T will be assumed to be *non-contracting*, i.e., such that for all x, y in the same element $Z \in \mathcal{Z}$, $d(Tx, Ty) \geq d(x, y)$.

Also $\mathcal{Z}$ will be assumed to be *generating*, i.e., $\lim_{n\to\infty} \operatorname{diam}(\mathcal{Z}^n) = 0$ where $\mathcal{Z}^n$ denotes the set of n-cylinders, i.e., the non-empty sets of the form

$$[A_0 \dots A_{n-1}] := A_0 \cap \dots \cap T^{-n+1}A_{n-1}$$

for $A_0, \dots, A_{n-1} \in \mathcal{Z}$.

Finally the *boundary of the partition*, $\partial\mathcal{Z} = \bigcup_{Z\in\mathcal{Z}} \partial Z$, will play an important role in our analysis. In particular, we shall assume "small boundary pressure" (see below), a fundamental condition which already appeared in [Bu2, Bu3, BPS].

A basic example is given by the *multidimensional β-transformations* [Bu0], i.e., maps T: $[0,1]^d \to [0,1]^d$, $T(x) = B.x \bmod \mathbb{Z}^d$ with B an expanding affine map on $\mathbb{R}^d$. An interesting choice of weight is the constant $g(x) = |\det B|^{-1}$.

These systems usually have plenty of invariant probability measures, some of them rather irrelevant such as those supported on periodic orbits. A classical way to select "interesting" measures is through the following variational principle [DGS]. One considers *equilibrium states*, i.e., invariant probability measures which maximize the *measure-theoretic pressure*:

$$h(\mu, T) - \int_Y \log g d\mu$$

($h(\mu, T)$ is the entropy of μ — see [DGS]).

In our *piecewise, multidimensional and expanding* setting, existence and uniqueness of these measures have been studied in [Bu2] and [BPS]. Furthermore, in [Bu2] such measures have been given the alternative and more "geometric" characterization of being exactly the invariant probability measures absolutely continuous w.r.t. a "conformal measure". Recall that a conformal measure is a not necessarily invariant, probability measure on X, such that the Jacobian of T w.r.t. this measure is equal to $e^{-P(X,T)}g^{-1}$ where $P(X,T)$ is the topological pressure defined below. This conformal measure can be given (like the Lebesgue measure if $g = |\det T'|^{-1}$) or constructed from the weight g [BPS].

In the example given above (multidimensional β-maps with $g = |\det B|^{-1}$), the conformal measure is just Lebesgue measure and thus the equilibrium states are the absolutely continuous invariant probability measures, which can indeed be considered interesting.

Statement of Results. To formulate the crucial "small boundary pressure" condition, we need first some definitions.

The **topological pressure** [DGS] of a subset S of X is

$$P(S, T) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{A\in \mathcal{Z}^n\\ \bar{A}\cap S\neq\emptyset}} g^{(n)}(A)$$

where $g^{(n)}(A) = \sup_{x\in A} g(x)g(Tx)\dots g(T^{n-1}x)$.

The **small boundary pressure** condition is

$$P(\partial\mathcal{Z}, T) < P(X, T).$$

This inequality is satisfied in many cases. In particular, if T is expanding and X is a Riemannian manifold and the weight is $|\det T'(x)|^{-1}$ or close to it, then it is satisfied: (i) in dimension 1, in all cases; (ii) in dimension 2, if T is piecewise real analytic [Bu3, T1]; (iii) in arbitrary dimension, for all piecewise affine T [T2] or for generic T [Bu1, Co]. See, however, [Bu4] for an expanding counter-example in dimension 2.

In our favorite example (multidimensional β-transformations, see above), the inequality is satisfied for arbitrary g as soon as

$$\frac{\sup g}{\inf g} < \left(\frac{\Lambda_-}{\Lambda_+}\right)^d \Lambda_+$$

with Λ_+, resp. Λ_-, the largest, resp. the smallest, modulus of the eigenvalues of the linear map corresponding to the β-transformation. Clearly, this is satisfied for $g = \text{const} = |\det B|^{-1}$.

0.1. THE EXPANDING AND HÖLDER CONTINUOUS CASE. Recall that a **conformal measure** for (X, T, g) is a probability measure ν such that $d\nu \circ T/d\nu = e^{-P(X,T)}g$.

MAIN THEOREM: *Let $(X, \mathcal{Z}, T, g)$ be a weighted piecewise invertible dynamical system. Assume that:*

- H1. *T is expanding, i.e., there is some $\lambda > 1$ such that for all x, y in the same element of $\mathcal{Z}$, $d(Tx, Ty) \geq \lambda \cdot d(x, y)$.*
- H2. *g is Hölder continuous with exponent γ and is positively lower bounded. Let*

$$K(f) = \max_{Z \in \mathcal{Z}} \sup_{x \neq y \in Z} \frac{|f(x) - f(y)|}{d(x, y)^\gamma}$$

where γ is some Hölder exponent of g.

- H3. *The boundary pressure is small: $P(\partial\mathcal{Z}, T) < P(X, T)$.*
- H4. *There is a conformal measure ν such that $T^n X \subset \text{supp}(\nu)$ (mod 0) for large enough n.*

Then, T admits a finite number of ergodic and invariant measures, $\mu_1, \ldots, \mu_r$, absolutely continuous w.r.t. ν with the following properties. Any invariant probability measure absolutely continuous w.r.t. ν is a convex combination of these measures. Each μ_i is exponentially mixing up to a period p_i, i.e., μ_i can be written $\frac{1}{n}\sum_{j=0}^{p_i-1} T_*^j \mu_i^0$ with $T_*^{p_i}\mu_i^0 = \mu_i^0$ and, for all $n \geq 1$,

$$\left| \int_X \varphi \circ T^{np_i} \cdot \psi d\mu_i^0 - \int_X \varphi d\mu_i^0 \int_X \psi d\mu_i^0 \right| \leq C \cdot \|\varphi\|_{C^\gamma(X)} \cdot \|\psi\|_{L^1(\mu_i^0)} \kappa^n$$

with constants $C < \infty$ and $\kappa < 1$ depending only on $(X, \mathcal{Z}, T, g)$, for any measurable functions $\varphi, \psi\colon X \to \mathbb{R}$ such that ψ is bounded and φ is γ-Hölder continuous.

We also obtain:

CENTRAL LIMIT THEOREM: *Under the same conditions, if φ is γ-Hölder continuous and satisfies $\int_X \varphi d\mu_i^0 = 0$, then setting*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_X \left(\sum_{k=0}^{n-1} \varphi \circ T^k \right)^2 d\mu_i^0$$

we have $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^k \Longrightarrow \mathcal{N}(0,\sigma)$, where $\Longrightarrow$ is the convergence in law and $\mathcal{N}(0,\sigma)$ is the normal distribution with mean zero and variance σ (the Dirac measure at 0 if $\sigma = 0$). Moreover, $\sigma = 0$ iff $\varphi = \psi - \psi \circ T$ for some $\psi \in L^2(\mu_i^0)$.

0.2. THE GENERAL CASE. The driving factor is the smoothness of the weight g as measured by the following sequence:

$$\omega_n(g) = \sup_{Z\in\mathcal{Z}^n} \sup_{x,y\in Z} \log \frac{g(x)}{g(y)}.$$

In fact, it is convenient to assume only that $\omega_n(g)$ is not less than the right hand side of the above equation.

Such a sequence defines the following functional space. It is the set of functions φ on X such that

$$\sup_{Z\in\mathcal{Z}^p} \sup_{x,y\in Z} |\varphi(x) - \varphi(y)| \le K \cdot \sum_{q>p} \omega_q(p).$$

We set $K(\varphi)$ to be the infimum of all numbers K such that the above equation holds for all $p \ge 1$. This condition together with this functional space were considered by B. Schmitt [Sc, KMS].

We may now state:

MAIN THEOREM (general version): *Let $(X, \mathcal{Z}, T, g)$ be a weighted piecewise invertible dynamical system. Assume that:*

H1. *T is non-contracting.*

H2. *g satisfies $\sum_{n\ge1} \omega_n(g) < \infty$ and that it is positively lower bounded.*

H3. *$P(\partial\mathcal{Z}, T) < P(X,T)$.*

H4. *There is a conformal measure ν such that $T^n X \subset \operatorname{supp}(\nu) \pmod 0$ for large enough n.*

Then, T admits a finite number of ergodic and invariant measures, $\mu_1, \dots, \mu_r$, absolutely continuous w.r.t. ν. Each μ_i is mixing up to a period p_i, i.e., μ_i can be written $\frac{1}{n} \sum_{j=0}^{p_i-1} T_*^j \mu_i^0$ with $T_*^{p_i} \mu_i^0 = \mu_i^0$ and, for all $n \ge 1$,

$$\left| \int_X \varphi \circ T^{np_i} \cdot \psi d\mu_i^0 - \int_X \varphi d\mu_i^0 \int_X \psi d\mu_i^0 \right| \le (\sup|\varphi| + K(\varphi)) \cdot \|\psi\|_{L^1(\mu_i^0)} u_n$$

for any measurable functions $\varphi, \psi: X \to \mathbb{R}$ *such that* ψ *is bounded and* $K(\varphi) < \infty$

The sequence $(u_n)_{n\geq 1}$ *depends on* (T, g) *and goes to zero with a speed which can be made explicit:*

1. *if* $\omega_n(g) = \mathcal{O}(\rho^n)$ *for some* $\rho < 1$, *then* $u_n = \mathcal{O}(\kappa^n)$ *for some* $\kappa < 1$;
2. *if* $\omega_n(g) = \mathcal{O}(n^{-\alpha})$ *for some* $\alpha > 1$, *then* $u_n = \mathcal{O}(n^{-(\alpha-1)})$;
3. *if* $\omega_n(g) = \mathcal{O}(e^{-n^{\alpha}})$ *for some* $0 < \alpha < 1$, *then* $u_n = \mathcal{O}(e^{-n^{\alpha-\varepsilon}})$ *for arbitrary* $\varepsilon > 0$.

We also obtain:

CENTRAL LIMIT THEOREM (general version): *Under the same conditions, if one also has* $\sum_{n\geq 1} u_n < \infty$, *for all* φ *such that* $K(\varphi) < \infty$ *and* $\int_X \varphi d\mu_i^0 = 0$, *setting*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int_X \left(\sum_{k=0}^{n-1} \varphi \circ T^k \right)^2 d\mu_i^0$$

we have $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^k \Longrightarrow \mathcal{N}(0, \sigma)$, *where* $\Longrightarrow$ *is the convergence in law and* $\mathcal{N}(0, \sigma)$ *is the normal distribution with mean zero and variance* σ *(the Dirac measure at* 0 *if* $\sigma = 0$*). Moreover,* $\sigma = 0$ *iff* $\varphi = \psi - \psi \circ T$ *for some* $\psi \in L^2(\mu_i^0)$.

OUTLINE OF THE PAPER. The rest of the paper is devoted to the proof of the above theorems.

First we define a Markov extension –or tower– $(\hat{X}, \hat{T})$ *à la* Hofbauer-Keller. Then we check that the return time $\hat{R}$ with respect to a slightly enlarged basis $\hat{X}_*$ of this tower satisfies an exponential estimate with rate $\exp -(P(X,T) - P(\partial\mathcal{Z}, T))$. Finally, we build from this Markov extension an abstract tower in the spirit of L.-S. Young [Y0]. We show that the a.c.i.m.'s on X lift to a.c.i.m.'s on this tower. From this, the theorems above follow from an abstract result which is proved in a companion paper [BuMa] dealing with the non-Markov case. In the expanding and Hölder continuous case, they also follow from a slight adaptation of [Y0].

COMMENTS.

- Our Main Theorem generalizes results well-known in the *globally expanding* case (see, e.g., [Bo]) or in the *piecewise expanding* one-dimensional setting (see [Ke1, LSV] and the references therein). The non-Hölder (piecewise invertible) case is new, even in the one-dimensional setting with $g = |\det T'|$ —we remark that the claim in [Go] that (in dimension 1 and with $g = |\det T'|^{-1}$) the decay of correlations is always exponential rests on a faulty lemma.

The special case $g = |\det T'|^{-1}$, with g Hölder continuous, has been considered in a multidimensional setting (under a slightly more restrictive condition than the small boundary pressure condition) in [Bu3, Sau].

- The assumption of small boundary pressure (H3) is natural. Indeed it reduces, for instance, in dimension 1, to a standard "spectral gap" condition. It also implies the conditions appearing in [Bu3, Sau] by a remark of M. Tsujii.
- The existence of a conformal measure (H4) is automatic in the case with $g = |\det T'|^{-1}$ (Lebesgue measure is enough). In the general case, it was proved in a similar setting by J. Buzzi, F. Paccaut and B. Schmitt [BPS].
- L. S. Young [Y2] has asked how one could *control the upper floors* of the tower for equilibrium states. We prove here that they are controlled in thermodynamical terms: their conformal measure decreases like the quantity $e^{-(P(X,T)-P(\partial\mathcal{Z},T))n}$ —see Proposition 1.1.
- By going to a Markov extension, we avoid spaces of discontinuous functions, i.e., spaces of functions with bounded *multi-dimensional variation.* This is already a significant technical gain in the case of Lebesgue measure ($g = |\det T'|^{-1}$) and seems necessary for more general weights as the classical functional spaces are no longer adapted and tailored ones (see [Ke1, Sau]) become too difficult to control ([P] studies, however, a promising functional space).
- Our results are deduced from similar statements about a tower extension in the spirit of L.-S. Young (see [Y0, Y1]). These statements, quoted at the end of our proof, are proved in [BuMa].

ACKNOWLEDGEMENT: The authors are grateful to *program ESF/PRODYN* which has partially supported the *International Conference on Dynamical Systems, Abbey of "La Bussière"* where most of this work was carried out. We thank the referees of a first version of this paper for their remarks.

1. The Markov extension

We define a Markov tower extension $(\hat{X}, \hat{T})$ in the vein of F. Hofbauer [Ho] and G. Keller [Ke2] and then prove that the pressure of the boundary provides an exponential estimate on the tail statistics of the return times to the basis.

1.1. CONSTRUCTION. Let $(X, \mathcal{Z}, T, g, \nu)$ be as in the Main Theorem. Let $\hat{X}_0 = \{(x, Z)\colon x \in Z \text{ and } Z \in \mathcal{Z}\}$. Recall that $Y = \bigcup_{Z\in\mathcal{Z}} Z$ is the domain of T.

For $x \in Y \cap T^{-1}Y$ and $A \subset X$, set

$$\hat{T}(x, A) = (Tx, T(A) \cap \mathcal{Z}[Tx])$$

where $\mathcal{Z}[x]$ is that element of $\mathcal{Z}$ which contains x.

Set $\hat{X} = \bigcup_{n \geq 0} \hat{T}^n(\hat{X}_0)$. Remark that

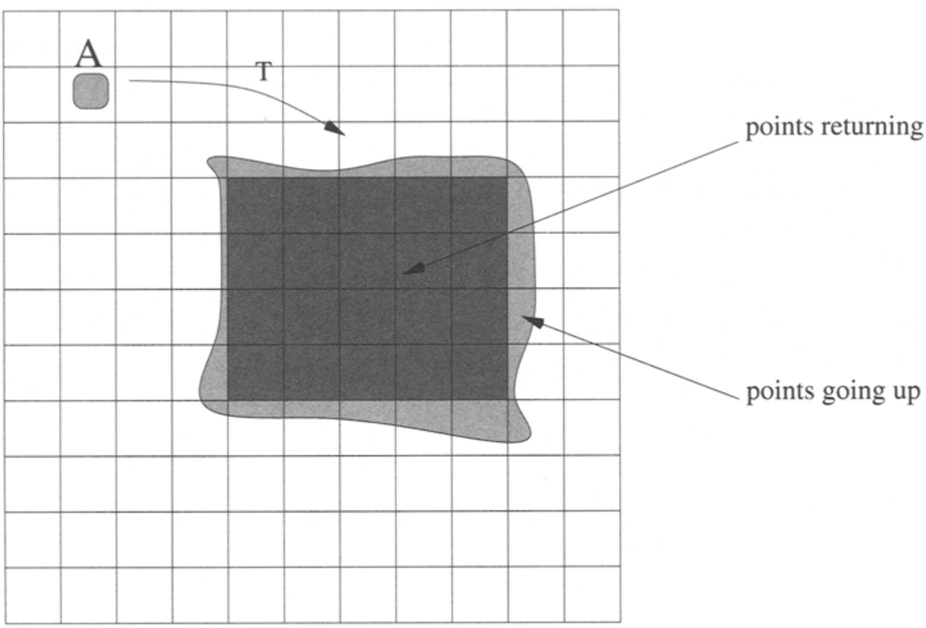
$$\hat{X} = \bigcup_{D \in \mathcal{D}} \hat{D} \text{ with } D \times \{D\} \quad \text{and} \quad \mathcal{D} = \{T^n Z : n \geq 0 \text{ and } Z \in \mathcal{Z}^{n+1}\}.$$

Thus, $\hat{X}$ is a countable, disjoint union of (sets naturally isomorphic to) subsets of X.

$\mathcal{D}$ has a natural graph structure: $D \to D'$ iff $D' = T(D) \cap Z$ for some $Z \in \mathcal{Z}$.

1.1. The Statistics of Returns. We shall consider returns to an enlarged basis of the Markov extensions: $\hat{X}_* = T^{N_*}\hat{X}_0$ for some N_*. $\hat{X}_*$ can also be seen as the disjoint union of $\mathcal{D}_*$, the collection of the sets $T_*^N Z$, $Z \in \mathcal{Z}^{N_*+1}$.

The figure below illustrates why points not too close to the boundary return to $\hat{X}_0$, explaining the idea behind the next proposition.



How points return to $\hat{X}_0$.
$\mathcal{Z}$ is a grid of small squares and we have drawn some set A and its image and pointed out the behavior of $\hat{T}(x, A)$ for $x \in A$.

PROPOSITION 1.1: *For each $\delta > 0$, if $N_* < \infty$ is large enough, then for*

$$\hat{X}_* = \hat{T}^{N_*}\hat{X}_0,$$

the return time to $\hat{X}_*$, $\hat{R}(\hat{x}) = \inf\{n \geq 1 : \hat{T}^n(\hat{x}) \in \hat{X}_*\}$, satisfies, for each $D \in \mathcal{D}$,

$$\nu(\{x \in D : \hat{R}((x, D)) > n\}) \leq \text{const} \cdot \exp[-n(P(X,T) - P(\partial\mathcal{Z}, T) - \delta)]$$

for some positive number const $< \infty$. *This number depends on D.*

Proof: Let $\delta > 0$ and $D \in \mathcal{D}$. Observe that D being of the form $T^k(A_0 \cap \cdots \cap T^{-k}A_k)$, $P(\partial D, T) \leq P(\partial\mathcal{Z}, T)$. Thus, for N large enough, for all $n \geq 0$,

$$(1) \qquad \sum_{\substack{Z \in \mathcal{Z}^n \\ \bar{Z} \cap (\partial D \cup \partial \mathcal{Z}) \neq \emptyset}} g^{(n)}(Z) \leq e^{\delta N} \cdot e^{(P(\partial\mathcal{Z},T)+\delta)n} \quad \text{and} \quad \binom{n}{[2n/N]} \leq e^{\delta n}.$$

$[\cdot]$ is the integer part.

Set

$$\varepsilon_1 = \min_{\Delta \in \{\partial\mathcal{Z}, \partial D\}} \{d(Z, \Delta) : 0 \leq n \leq N+1,\ Z \in \mathcal{Z}^n \text{ such that } \overline{Z} \cap \Delta = \emptyset\}.$$

$\varepsilon_1 > 0$ as each $\mathcal{Z}^n$ is finite. Obviously,

$$d(x, \partial\mathcal{Z}[x]) < \varepsilon_1 \Longrightarrow \forall 0 \leq n \leq N+1\ \overline{\mathcal{Z}^n[x]} \cap \partial\mathcal{Z} \neq \emptyset$$

as soon as $\mathcal{Z}^n[x]$ is well-defined.

Let $\varepsilon_2 > 0$ be such that for all $x \in X$ there is a connected $\Gamma \subset X$ with $B(x, \varepsilon_2) \subset \Gamma \subset B(x, \varepsilon_1)$. This exists because of the local connectedness and the compactness of X.

For $x \in D$ and $n \geq 0$, let

$$\rho_n(x) := \sup\{r > 0 : B(T^n x, r) \subset T^n(D \cap \mathcal{Z}^{n+1}[x])\}$$

and set

$$R(x) := \min\{n \geq 0 : \rho_n(x) \geq \varepsilon_2\}.$$

Choose $N_* < \infty$ so large that the diameter of the partition $\mathcal{Z}^{N_*+1}$ is less than ε_2. Let us observe that

$$\hat{R}((x, D)) \leq R(x) + N_*.$$

Indeed, as

$$\mathcal{Z}^{N_*+1}[T^{R(x)}x] \subset B(T^{R(x)}x, \varepsilon_2) \subset T^{R(x)}(D \cap \mathcal{Z}^{R(x)+1}[x])$$

we have

$$\begin{aligned}\hat{T}^{R(x)+N_*}(x, D) &= (T^{R(x)+N_*}x, T^{R(x)+N_*}(\mathcal{Z}^{R(x)+N_*+1}[x] \cap D)) \\ &= (T^{R(x)+N_*}x, T^{N_*}(\mathcal{Z}^{N_*+1}[T^{R(x)}x] \cap T^{R(x)}(D \cap \mathcal{Z}^{R(x)+1}[x]))) \\ &= (T^{R(x)+N_*}x, T^{N_*}\mathcal{Z}^{N_*+1}[T^{R(x)}x]) \\ &= \hat{T}^{N_*}(T^{R(x)}x, \mathcal{Z}[T^{R(x)}x]) \in T^{N_*}\hat{X}_0 =: \hat{X}_*.\end{aligned}$$

It is therefore enough to prove the exponential estimate for $\nu(\{x \in D : R(x) > n\})$.

CLAIM: *For every $x \in D$ such that $R(x) > n$, there exists a finite sequence of times $n_0 = 0 < n_1 < \cdots < n_r = n$ such that, for all $0 \le i < r$,*

$$r \le \frac{2n}{N} + 1, \tag{2}$$

$$\overline{\mathcal{Z}^{n_{i+1}-n_i}[T^{n_i}x]} \cap (\partial Z \cup \partial D) \neq \emptyset, \tag{3}$$

where N was defined in (1). An interval $[n_i, n_{i+1})$ satisfying eq. (3) is said to be a **shadowing interval**.

Let us see that the claim implies the proposition. Let

$$C = \sup_n \sup_{Z \in \mathcal{Z}^n} \sup_{x,y \in Z} \frac{g^{(n)}(x)}{g^{(n)}(y)} \le \sup_n \exp \sum_{k=1}^{n} \omega_k(g) \le \exp \sum_{k \ge 1} \omega_k(g) < \infty.$$

We compute

$$\begin{aligned}\nu(\{R > n\}) &\le \sum_{\substack{Z \in \mathcal{Z}^n \\ \exists x \in Z \text{ s.t. } R(x) > n}} \nu(Z) \\ &\le \sum_{\substack{Z \in \mathcal{Z}^n \\ \exists x \in Z \text{ s.t. } R(x) > n}} C \cdot e^{-nP(X,T)} g^{(n)}(x) \cdot \nu(T^n Z)\end{aligned}$$

(we used the fact that ν is conformal). Hence the claim implies

$$\begin{aligned}\nu(\{R > n\}) &\le C \cdot e^{-nP(X,T)} \sum_{r=1}^{(2n/N)+1} \sum_{n_0,\dots,n_r} \prod_{i=0}^{r-1} \sum_{\substack{Z \in \mathcal{Z}^{n_{i+1}-n_i} \\ \bar{Z} \cap (\partial Z \cup \partial D) \neq \emptyset}} g^{(n_{i+1}-n_i)}(Z) \\ &\le C \cdot e^{-nP(X,T)} \left(\frac{2n}{N} + 1\right) \binom{n}{[\frac{2n}{N}]} e^{\delta N \frac{2n}{N}} e^{n(P(\partial \mathcal{Z}, T)+\delta)} \\ &\le C \cdot e^{-n(P(X,T) - P(\partial \mathcal{Z}, T) - 5\delta)}\end{aligned}$$

which is the statement of the Proposition (up to substituting $\delta/5$ for δ). Thus, we only need prove the claim by building the required times n_i.

The claim will follow from the following combinatorial Lemma.

LEMMA 1.2: *Let $n \geq 1$ and let C be a cover of $[0,n)$ by (integer) subintervals $[a,b)$ which are* long *in the sense that $b \geq \min(a+N, n)$. Then there exists a partition of $[0,n)$ into at most $2n/N+1$ subintervals, each of which is the beginning of some interval in C.*

Deferring the proof of this lemma, we just have to see that it may be applied, i.e., $[0,n)$ is covered by long shadowing intervals. We prove that an arbitrary integer $k \in [0,n)$ is contained in a long shadowing interval.

As $k < n \leq R(x)$, $B(T^k x, \varepsilon_2) \not\subset T^k(D \cap \mathcal{Z}^{p+1}[x])$. By the choice of ε_2, there exists a connected set Γ with $B(T^k x, \varepsilon_2) \subset \Gamma \subset B(T^k x, \varepsilon_1)$. Hence Γ must meet the boundary of $T^k \mathcal{Z}^{k+1}[x]$. Let u be a point in the intersection. Obviously, $u = \overline{T^{k-\ell}|\mathcal{Z}^{k-\ell+1}[T^\ell x]}(v)$ for some $0 \leq \ell \leq k$ and $v \in \partial\mathcal{Z}$.

$[\ell, k+1)$ is a shadowing interval. If it is long, there is nothing else to show. Thus, we assume that it is not long: $k - \ell \leq N$.

As the map is non-contracting, $d(T^\ell x, v) \leq d(T^k x, u) < \varepsilon_1$. Hence,

$$d(\mathcal{Z}^m[T^\ell x], \partial\mathcal{Z}) < \varepsilon_1 \quad \text{with } m = \min(N, n-\ell).$$

By the choice of ε_1, this implies $\overline{\mathcal{Z}^m[T^\ell x]} \cap \partial\mathcal{Z} \neq \emptyset$. $[\ell, \ell+m) \ni k$ is the long shadowing interval we were looking for. This concludes the proof of the claim and of the proposition, modulo the proof of Lemma 1.2, to which we now turn. ∎

Proof of Lemma 1.2: The sought-for partition will be

$$[n_1, n_2), [n_2, n_3), \ldots, [n_{i_{\max}-1}, n_{i_{\max}}).$$

Let $n_1 = 0$ and m_1 be maximum with $[0, m_1) \in C$. We define the sequences n_i and m_i inductively.

Assume n_{i-1} and m_{i-1} are defined. If $n_{i-1} = n$, set $i_{\max} = i-1$ and terminate. If $n_{i-1} < m_{i-1} = n$, set $i_{\max} = i$, $n_{i_{\max}} = m_{i-1}$ and terminate. Otherwise, consider the intervals in C containing m_{i-1} (there exists at least one such interval containing m_{i-1} by the covering assumption). Pick one with the right endpoint as far to the right as possible. Denoting this interval by $[a,b)$, set $n_i := a$ and $m_i := b$. This completes the definition of the n_i's and m_i's.

Observe that for all $1 \leq i \leq i_{\max}$, $m_i - n_i \geq N$ as $[n_i, m_i) \in C$, except if $m_i = n$.

Also, for $1 < i < i_{\max}$, $n_{i+1} > m_{i-1}$, because otherwise $[n_{i+1}, m_{i+1}) \ni m_{i-1}$ and $m_i < m_{i+1}$ would contradict the choice of m_i.

Therefore, for all $0 < i < i_{\max}$,

$$n_{i+1} > m_{i-1} \geq n_{i-1} + N.$$

This proves the Lemma. ∎

2. The inducing tower

Working from the previous Markov extension $\hat{X}$, we define another tower describing the map induced by $\hat{T}$ on $\hat{X}_*$ following L.-S. Young's philosophy [Y1].

Let $\Delta_0 = \hat{X}_*$ be the enlarged basis of the Markov tower previously defined with N_* large enough both w.r.t. Proposition 1.1 (with $\delta < P(X,T) - P(\partial \mathcal{Z}, T)$) and hypothesis (H4) (so that $T^{N_*}X \subset \operatorname{supp} \nu$).

We shall write x for a point of $\Delta_0 \subset X \times \mathcal{D}$.

Let $(\Delta_{0,j})_{j \in \mathbb{N}}$ be the partition of Δ_0 obtained by dividing it first into level sets $\{x : \hat{R}(x) = r\}$ and then according to $\mathcal{Z}^{r+1}$ on the rth level set, i.e., $(\Delta_{0,j})_{j \in \mathbb{N}}$ is the coarsest partition of Δ_0 such that $\hat{R}(x)$ and $\mathcal{Z}_{\hat{R}(x)}[x]$ are constant on atoms.

We define

$$\Delta_\ell = \{(x, \ell)/x \in \Delta_0 \text{ and } \hat{R}(x) > \ell\}.$$

The tower $\Delta \subset X \times \mathcal{D} \times \mathbb{N}$ is the disjoint union of the Δ_ℓ's. We will denote by $\Delta_{\ell,j} \subset \Delta_\ell$ the set of (x, ℓ) such that $(x, 0)$ belongs to $\Delta_{0,j}$. The $\Delta_{\ell,j}$'s for $\ell \in \mathbb{N}$, $j \in \mathbb{N}$, form a partition of Δ which we denote by $\mathcal{P}$.

Let F: $\Delta \longrightarrow \Delta$ be defined in the following way:

$$\begin{cases} F(x,\ell) = (x, \ell+1) & \text{if } \ell + 1 < \hat{R}(x), \\ = (\hat{T}^{\hat{R}(x)}x, 0) & \text{otherwise.} \end{cases}$$

We remark that we do not assume that the images $F^R\Delta_{0,j}$ are the whole basis Δ_0. That is why we have to introduce the partition $\{B_1, \ldots, B_p\}$ of Δ_0 generated by $\{F^R\Delta_{0,j} : j \in \mathbb{N}\}$. It is finite because each $F^R\Delta_{0,j}$ corresponds to some union of elements of $\mathcal{D}_*$, which is finite.

Finally, we lift the probability measure ν on X to a measure $\hat{\nu}$ on Δ defined by

$$\hat{\nu}(S \times \{D\} \times \{\ell\}) = \nu(S).$$

Since Δ_0 is a finite union of $\widehat{D}$, $D \in \mathcal{D}$, $\hat{\nu}(\Delta_0) < \infty$. Also, by the choice of N_*, $\operatorname{supp}(\hat{\nu}) = \Delta$.

We define the following metric on Δ:

$$d_0(x,y) = C \cdot \exp \sum_{j>s(x,y)} \omega_j(g),$$

where $s(x,y) = \min\{k \geq 0 : F^k(x), F^k(y) \text{ are not in the same element of } \mathcal{P}\}$ and $C = \exp \sum_{j\geq 1} \omega_j(g)$. This last constant is introduced to have property (A.III) below.

Let us summarize the crucial properties of the tower.

(A.I) **Exponentially small upper floors.** Because of Proposition 1.1,

$$\hat{\nu}(\{x \in \Delta_0 / \hat{R}(x) > \ell\}) \leq \text{const} \cdot \gamma^\ell$$

for some $0 < \gamma < 1$. In particular, $\hat{\nu}(\Delta) = \sum_\ell \hat{\nu}(\Delta_\ell) \leq \text{const} \sum_\ell \gamma^\ell < \infty$. Moreover, the support of $\hat{\nu}$ is Δ: this follows from our construction and the fact that the support of ν contains $T^{N_*}X$. In what follows, we assume that $\hat{\nu}$ has been normalized, i.e., $\hat{\nu}(\Delta) = 1$.

(A.II) **Generating partition.** The partition $\mathcal{P}$ generates under F, i.e., the partition $\bigvee_{n=0}^\infty F^{-n}\mathcal{P}$ is the partition into points. In particular, d_0 defines a metric on Δ.

(A.III) **Bounded distortion.** Let JF be the Jacobian of F with respect to $\hat{\nu}$. Obviously, $JF(x,\ell) = 1$ if $\ell + 1 < \hat{R}(x)$ and $JF(x, \hat{R}(x) - 1) = g_{\hat{R}(x)}(x)$. If x, y are in the same B_j, we can define their paired pre-images x' and y' as follows: $F^n x' = x$, $F^n y' = y$ and $F^k(x')$ and $F^k(y')$ belong to the same element of $\mathcal{P}$ for $0 \leq k < n$. We have

$$\left| \frac{JF^n(x')}{JF^n(y')} - 1 \right| \leq d_0(x,y).$$

Indeed, recalling that $C = \sum_{j\geq 1} \omega_j(g) \geq \log \frac{JF^n(x')}{JF^n(y')}$,

$$\left| \frac{JF^n(x')}{JF^n(y')} - 1 \right| \leq C \cdot \left| \log \frac{JF^n(x')}{JF^n(y')} \right| \leq C \cdot \sum_{j \geq n+s(x,y)} \omega_j(n).$$

(A.IV) **Markov properties.** $\{F^{\hat{R}}(\Delta_{0,j}) : j \geq 0\}$ is finite and consists only of sets with positive measure. Moreover, each $F^{\hat{R}}\Delta_{0,j}$ is a union of some $\Delta_{0,p}$.

2.1. Lifting a.c.i.m.'s. To analyze an arbitrary, ergodic a.c.i.m. on X we first have to lift it to the inducing tower Δ:

PROPOSITION 2.1: *If μ is an ergodic ν-a.c.i.m. on X, then there exists an ergodic $\hat{\nu}$-a.c.i.m. $\hat{\mu}$ on Δ projecting to μ by the semi-conjugacy $\pi : \Delta \to X$, $\pi(x, D, \ell) = T^\ell x$.*

To prove this, we begin by finding a not necessarily invariant, probability measure $\hat{\mu}_0$ on Δ with projection on X absolutely continuous w.r.t. μ. We set $\hat{\mu}_0 = 1_{\hat{X}_* \cap \pi^{-1}(S)} \cdot \hat{\nu}$, where S is invariant and such that $\mu(S) = 1$ and $\mu'(S) = 0$ for all other ergodic ν-a.c.i.m.'s (there can be only countably many measures which are pairwise singular and absolutely continuous w.r.t. a given measure).

We have to check that this $\hat{\mu}_0$ can be normalized. Because of (A.I), $\pi_*(\hat{\nu}) \leq C \cdot \nu(X) < \infty$. We have to see that $\hat{\mu}_0$ is not zero, i.e., $\hat{\nu}(\hat{X}_* \cap \pi^{-1}(S)) > 0$. But this follows from the invariance of S and the fact that the $\hat{\nu}$-a.e. point of $\hat{X}$ eventually enters $\hat{X}_*$ by Proposition 1.1.

Thus we can normalize $\hat{\mu}_0$.

Now, we claim that we can take $\hat{\mu}$ to be any weak* limit point, say $\hat{\mu}$, of the sequence $\frac{1}{n}\sum_{k=0}^{n-1} F_*^k \hat{\mu}_0$.

Clearly $\hat{\mu}$ is F-invariant. Let us show that $\hat{\mu}$ is $\hat{\nu}$-absolutely continuous. For this we consider the sequence of the corresponding densities. Introducing the transfer operator for $(F, \hat{\nu})$,

$$\mathcal{L}f(x) = \sum_{y \in F^{-1}(x)} \frac{f(y)}{JF(y)}$$

where JF is the Jacobian of F w.r.t. $\hat{\nu}$, these densities are $\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}^k d\hat{\mu}_0/d\hat{\nu}$ (recall that $d\hat{\mu}_0/d\hat{\nu} = C$). The absolute continuity will therefore follow from the

CLAIM: *There is a constant $K < \infty$ such that for all $x \in \Delta$, $n \geq 0$, $(\mathcal{L}^n 1)(x) \leq K$.*

The claim together with $\hat{\nu}(\Delta) = 1$ allows the application of the Lebesgue dominated convergence theorem to see that the μ-absolute continuity of the projections of the measures $F_*^k \hat{\mu}_0$ passes to the limit $\hat{\mu}$. For the same reason, $\hat{\mu}$ is a probability measure. Finally, as $\pi_*\hat{\mu} << \nu$ and $\pi_*\hat{\mu}(S) = 1$, $\pi_*\hat{\mu} << \mu$. By ergodicity, this implies $\pi_*\hat{\mu} = \mu$, proving the Proposition, except for the claim.

We need some facts. Recall that $\mathcal{P}$ is the partition of Δ obtained by translation from the partition of Δ_0 according to the return time $\hat{R}$ and $\mathcal{Z}^{\hat{R}}$-itinerary. For $k \in \mathbb{N}$ and $x \in \Delta$, $C_k(x)$ denotes the $k, \mathcal{P}$-cylinder which contains x.

LEMMA 2.2: *There exists $C < \infty$ such that for any $\ell \in \mathbb{N}$, $x \in \Delta_\ell$ and $k \in \mathbb{N}$ with $F^k x \in \Delta_0$,*

$$C^{-1}\hat{\nu}(C_k(x)) \leq \frac{1}{JF^k(x)} \leq C\hat{\nu}(C_k(x)).$$

Proof: Let $x \in \Delta_\ell$ such that $F^k(x) \in \Delta_0$. The Markov property and the large image property (A.IV) imply that $\hat\nu(F^k C_k(x)) \geq \eta$, a positive constant. The bounded distortion property (A.III) gives

$$C^{-1}\frac{\hat\nu(C_k(x))}{\hat\nu(F^k C_k(x))} \leq \frac{1}{JF^k(x)} \leq C\frac{\hat\nu(C_k(x))}{\hat\nu(F^k C_k(x))},$$
$$C^{-1}\frac{\hat\nu(C_k(x))}{1} \leq \frac{1}{JF^k(x)} \leq C\frac{\hat\nu(C_k(x))}{\eta}.$$

The Lemma is proved. ∎

We now prove the claim.

The upper bound $\mathcal{L}^n \mathbf{1} \leq K$ follows from Lemma 2.2, writing

$$\mathcal{L}^n \mathbf{1}(x) = \sum_{x' \in F^{-n}x} \frac{1}{JF^n(x')} \leq C \sum_{x' \in F^{-n}x} \hat\nu(C_n(x')) \leq C \cdot \hat\nu(\hat X). \tag{1}$$

This concludes the proof of the Lemma.

3. Proof of the Theorems

Let us first quote a theorem with two corollaries that give our Main Theorem and Central Limit Theorem for an abstract tower. These are proved in a companion paper which considers a more general setting, in particular non-Hölder smoothness. However, the reader may remark that in the Hölder setting, these facts essentially reduce to L.-S. Young's abstract results [Y1], after taking care of the following technicalities:

- We have a "Markov rather than a Bernoulli picture": the returns are not to Δ_0, but to finitely many elements $B_1, \ldots, B_p \subset \Delta_0$.
- Our return times are not lower bounded by an arbitrary *a priori* constant.
- The tower is not aperiodic.

THEOREM 3.1 ([BuMa]): *Let $(\Delta, F, \hat\nu, d_0)$ be a tower system satisfying (A.I–IV). Let $\mathcal{L}$ be the transfer operator associated to it.*

Then, if $\hat\mu$ is a $\hat\nu$-a.c.i.m., there exist an integer $1 \leq p < \infty$, a decomposition $\Delta = \bigcup_{k=0}^{p-1} T^k \Delta_*$ (mod $\hat\mu$) ($F^p(\Delta_*) = \hat X_*$) and a function $h_* \in L^\infty(\Delta_*)$ such that, for any $\varphi : \Delta_* \to \mathbb{R}$ bounded with $K(\varphi) < \infty$,

$$\|\mathcal{L}^n \varphi - \hat\mu_*(\varphi) \cdot h_*\|_\infty \leq (\|\varphi\|_\infty + K(\varphi)) \cdot u_n$$

for all $n \geq 0$. Here $\hat\mu_* = p \cdot \hat\mu \cdot 1_{\Delta_*}$, $K(\varphi)$ is the supremum of the Lipschitz constants, w.r.t. d_0, of the restrictions $\varphi|\Delta_{i,j}$.

The sequence $(u_n)_{n\geq 1}$ depends on $(F, \hat{\nu})$ and goes to zero with a speed which can be made explicit:

1. *if $\omega_n(g) = \mathcal{O}(\rho^n)$ for some $\rho < 1$, then $u_n = \mathcal{O}(\kappa^n)$ for some $\kappa < 1$;*
2. *if $\omega_n(g) = \mathcal{O}(n^{-\alpha})$ for some $\alpha > 1$, then $u_n = \mathcal{O}(n^{-(\alpha-1)})$;*
3. *if $\omega_n(g) = \mathcal{O}(e^{-n^\alpha})$ for some $0 < \alpha < 1$, then $u_n = \mathcal{O}(e^{-n^{\alpha-\varepsilon}})$ for arbitrary $\varepsilon > 0$.*

COROLLARY 3.2 ([BuMa]): *In the same situation, we have that for all $\varphi, \psi \in L^\infty(\Delta_*)$ with $K(\varphi) < \infty$,*

$$\left| \int_\Delta \varphi \circ F^n \cdot \psi d\hat{\mu} - \int \Delta\varphi d\hat{\mu} \int_\Delta \psi d\hat{\mu} \right| \leq (\|\varphi\|_\infty + K(\varphi))\|\psi\|_{L^1(\hat{\mu}_*)} \cdot u_n$$

for all $n \geq 0$ for the same sequence $(u_n)_{n\geq 1}$ as above.

COROLLARY 3.3 ([BuMa]): *Assuming additionally that $\sum_{n\geq 1} u_n < \infty$, we have the Central Limit Theorem for all functions φ bounded and d_0-Lipschitz.*

To deduce our Main Theorem and our Central Limit Theorem, it is enough to first lift the ergodic ν-a.c.i.m. μ to Δ using Proposition 2.1 and make the following remark. If $\varphi, \psi \colon X \to \mathbb{R}$ are as in the Main Theorem, then $\varphi \circ \pi$ is γ-Hölder continuous and $\psi \circ \pi \in L^\infty$. Finally, note that the integrals over Δ involving these lifted functions are equal to the integrals in the Main Theorem. For instance,

$$\int_\Delta \varphi \circ \pi \cdot \psi \circ \pi \circ F^n d\hat{\mu} = \int_\Delta (\varphi \cdot \psi \circ T^n) \circ \pi d\hat{\mu} = \int_X \varphi \cdot \psi \circ T^n d\mu$$

by $\pi \circ F = T \circ \pi$ and $\pi_* \hat{\mu} = \mu$.

References

[Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Mathematics **470**, Springer-Verlag, Berlin, 1975.

[Bu0] J. Buzzi, *Intrinsic ergodicity of affine maps in* $[0,1]^d$, Monatshefte für Mathematik **124** (1997), 97–118.

[Bu1] J. Buzzi, *Absolutely continuous invariant measures for generic piecewise affine and expanding maps,* International Journal of Chaos and Bifurcation **9** (1999), 1743–1750.

[Bu2] J. Buzzi, *Thermodynamical formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states*, in *Smooth Ergodic Theory and its Applications (Seattle, WA, 1999)*, Proceedings of Symposia in Pure Mathematics **69**, American Mathematical Society, Providence, RI, 2001, pp. 749–783.

[Bu3] J. Buzzi, *Absolutely continuous invariant probability measures for arbitrary expanding piecewise* **R**-*analytic mappings of the plane*, Ergodic Theory and Dynamical Systems **20** (2000), 697–708.

[Bu4] J. Buzzi, *No or infinitely many a.c.i.p. for piecewise expanding C^r maps in higher dimensions*, Communications in Mathematical Physics **222** (2001), 495–501.

[BuKe] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps*, Ergodic Theory and Dynamical Systems **21** (2000), 689–716.

[BuMa] J. Buzzi and V. Maume-Deschamps, *Decay of correlations for non-Hölderian towers*, Preprint.

[BPS] J. Buzzi, F. Paccaut and B. Schmitt, *Conformal measures for multidimensional expanding maps*, Preprint, Laboratoire de Topologie #196 (Dijon, 1999) (see `http://www.math.polytechnique.fr/cmat/buzzi`).

[Co] W. Cowieson, *Piecewise smooth expanding maps in $\mathbb{R}^d$*, Ph.D. Thesis, U.C.L.A., Berkeley (see `http://math.usc.edu/~cowieson`).

[DGS] M. Denker, C. Grillenberger and K. Sygmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics **527**, Springer, Berlin, 1976.

[Go] P. Góra, *Properties of invariant measures for piecewise expanding one-dimensional transformations with summable oscillations of derivative*, Ergodic Theory and Dynamical Systems **14** (1994), 475–492.

[GB] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for piecewise expanding C^2 transformation in R^N*, Israel Journal of Mathematics **67** (1989), 272–286.

[Ho] F. Hofbauer, *Piecewise invertible dynamical systems*, Probability Theory and Related Fields **72** (1986), 359–386.

[HK] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Mathematische Zeitschrift **180** (1982), 119–140.

[Ke1] G. Keller, *Generalized bounded variation and applications to piecewise monotonic transformations*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **69** (1985), 461–478.

[Ke2] G. Keller, *Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems*, Transactions of the American Mathematical Society **314** (1989), 433–497.

[KMS] A. Kondah, V. Maume and B. Schmitt, *Vitesse de convergence vers l'état d'équilibre pour des dynamiques markoviennes non höldériennes*, Annales de l'Institut Henri Poincaré. Probabilités et Statistiques **33** (1997), 675–695.

[LY] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Transactions of the American Mathematical Society **186** (1973), 481–488.

[LSV] C. Liverani, B. Saussol and S. Vaienti, *Conformal measure and decay of correlations for covering weighted systems*, Ergodic Theory and Dynamical Systems **18** (1998), 1399–1420.

[P] F. Paccaut, *Propriétés statistiques de systèmes dynamiques non markoviens*, Thèse de l'université de Bourgogne, Juin 2000.

[R] M. Rychlik, *Bounded variation and invariant measures*, Studia Mathematica **76** (1983), 69–80.

[Sau] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel Journal of Mathematics **116** (2000), 223–248.

[Sc] B. Schmitt, *Existence of acip for expanding maps of the interval*, Dynamical Systems and Ergodic Theory, Banach Center Publications, Warszawa, 1989.

[T1] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Communications in Mathematical Physics **208** (2000), 605–622.

[T2] M. Tsujii, *Absolutely continuous invariant measures for expanding piecewise linear maps*, Inventiones Mathematicae **143** (2001), 349–373.

[T3] M. Tsujii, *Piecewise Expanding maps on the plane with singular ergodic properties*, Ergodic Theory and Dynamical Systems **20** (2000), 495–501.

[Y0] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Annals of Mathematics **147** (1998), 585–650.

[Y1] L.-S. Young, *Return times and rates of mixing*, Israel Journal of Mathematics **110** (1999), 153–188.

[Y2] L.-S. Young, *Decay of Correlations*, talk given at the International Conference on Dynamical Systems, La Bussière, July 1999.